

Carleman Estimate for Stochastic Parabolic Equations and Inverse Stochastic Parabolic Problems*

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Abstract

In this paper, we establish a global Carleman estimate for stochastic parabolic equations. Based on this estimate, we solve two inverse problems for stochastic parabolic equations. One is concerned with a determination problem of the history of a stochastic heat process through the observation at the final time T , for which we obtain a conditional stability estimate. The other is an inverse source problem with observation on the lateral boundary. We derive the uniqueness of the source.

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1 Introduction

In this paper, we solve two different inverse problems for stochastic parabolic equations by means of establishing a global Carleman estimate. To begin with, we introduce some notations.

Let $T > 0$, $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with a C^2 boundary Γ . Put

$$Q \triangleq (0, T) \times G, \quad \Sigma \triangleq (0, T) \times \Gamma.$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space on which a one dimensional standard Brownian motion $\{B(t)\}_{t \geq 0}$ is defined. Let H be a Banach space. Denote by $L^2_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0, T; H)}) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0, T; H)$ the Banach

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space consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted bounded processes; by $L^2_{\mathcal{F}}(\Omega; C([0, T]; H))$ the Banach space consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes $X(\cdot)$ satisfying that $\mathbb{E}(|X(\cdot)|^2_{C([0, T]; H)}) < \infty$, with the canonical norm.

Throughout this paper, we make the following assumptions on the coefficients

$$b^{ij} : \Omega \times Q \rightarrow \mathbb{R}^{n \times n}, \quad (i, j = 1, 2, \dots, n) :$$

(H1) $b^{ij} \in L^2_{\mathcal{F}}(\Omega; C^1([0, T]; W^{2, \infty}(G)))$ and $b^{ij} = b^{ji}$;

(H2) There is a constant $\sigma > 0$ such that

$$\sum_{i, j=1}^n b^{ij}(\omega, t, x) \xi^i \xi^j \geq \sigma |\xi|^2, \quad (\omega, t, x, \xi) \equiv (\omega, t, x, \xi^1, \dots, \xi^n) \in \Omega \times Q \times \mathbb{R}^n. \quad (1.1)$$

Let

$$a_1 \in L^\infty_{\mathcal{F}}(0, T; L^\infty(G; \mathbb{R}^n)), \quad a_2 \in L^\infty_{\mathcal{F}}(0, T; L^\infty(G)) \quad a_3 \in L^\infty_{\mathcal{F}}(0, T; W^{1, \infty}(G)),$$

$$f \in L^2_{\mathcal{F}}(0, T; L^2(G)) \quad \text{and} \quad g \in L^2_{\mathcal{F}}(0, T; H^1(G)).$$

Let

$$r_1 = |a_1|^2_{L^\infty_{\mathcal{F}}(0, T; L^\infty(G; \mathbb{R}^n))} + |a_2|^2_{L^\infty_{\mathcal{F}}(0, T; L^\infty(G))} + |a_3|^2_{L^\infty(0, T; W^{1, \infty}(G))} + 1.$$

Consider the following stochastic parabolic equation:

$$\begin{cases} dy - \sum_{i, j=1}^n (b^{ij} y_{x_i})_{x_j} dt = [(a_1, \nabla y) + a_2 y] dt + a_3 y dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma. \end{cases} \quad (1.2)$$

Here $y_{x_i} = \frac{\partial y}{\partial x_i}$.

The first inverse problem is concerned with the following problem:

Stochastic parabolic equation backward in time: Let $0 \leq t_0 < T$. Determine $y(\cdot, t_0)$, P -a.s. from $y(\cdot, T)$.

For deterministic parabolic equations, such kind of problem has lots of applications in the mathematical physics (e.g. [1]) and is studied extensively (see [24] for a nice survey). Generally speaking, the backward (stochastic) parabolic equation is ill-posed. Small errors in the measuring of the terminal data may cause huge deviations in final results, that is, there is no stability in this problem. Fortunately, if we assume a priori bound for $y(0)$ (such assumption is reasonable from a practical viewpoint), then we can regain the stability in some sense. The concept of conditional stability is used to describe such kind of stability. In general framework, the conditional stability problem can be formulated as follows:

Let $t_0 \in [0, T)$, $\alpha \geq 0$ and $M > 0$. Put

$$U_{M, \alpha} \triangleq \{f \in L^2(\Omega, P, \mathcal{F}_0; H^\alpha(G)) : |f|_{L^2(\Omega, P, \mathcal{F}_0; H^\alpha(G))} \leq M\}.$$

Can we choose a function $\beta \in C[0, +\infty)$ satisfies the following properties:

$$\begin{cases} 1. \beta \geq 0 \text{ and } \beta \text{ is strictly increasing;} \\ 2. \lim_{\eta \rightarrow 0} \beta(\eta) = 0; \\ 3. \beta(|y(t_0)|_{L^2(\Omega, \mathcal{F}_{t_0}, P; L^2(G))}) \leq \beta(|y(T)|_{L^2(\Omega, \mathcal{F}_T, P; L^2(G))}). \end{cases}$$

In this paper, we obtain the following result for the above conditional stability problem.

Theorem 1.1 *Let $y \in L^2_{\mathcal{F}}(\Omega; C([0, T]; H^1_0(G)))$ solve equation (1.2), then there exists a constant $\theta \in (0, 1)$ such that*

$$\mathbb{E}|y(t_0)|_{L^2(G)} \leq C |y|_{L^2_{\mathcal{F}}(0, T; L^2(G))}^{1-\theta} \mathbb{E}|y(T)|_{H^1(G)}^{\theta}. \quad (1.3)$$

Remark 1.1 *In deterministic setting, a stronger result for the above conditional stability problem was obtained in [20]. In [20], the authors study the following equation:*

$$\begin{cases} y_t - \Delta y = by & \text{in } Q, \\ y = 0 & \text{on } \Sigma. \end{cases} \quad (1.4)$$

Here b is a suitable function. With the assumption that G is convex, they get

$$|y(0)|_{L^2(G)}^2 \leq C \exp\left(\frac{|y(0)|_{L^2(G)}}{|y(0)|_{H^{-1}(G)}}\right) |y(T)|_{L^2(G_0)}^2. \quad (1.5)$$

Here G_0 is any open subset of G . Compared with Theorem 1.1, only $|y(T)|_{L^2(G_0)}^2$ is involved in the right hand side of the inequality. However, they need G to be convex. They prove this result by employing some special frequency functions, which are first constructed for proving the doubling property of the solution of heat equations. However, since the solution of equation (1.2) is non-differentiable with respect to t , it seems that their method cannot be easily adopted to solve our problem.

As a consequence of Theorem 1.1, we obtain a backward uniqueness for equation (1.2).

Corollary 1.1 *Assume that $y \in L^2_{\mathcal{F}}(\Omega; C([0, T]; H^1_0(G)))$ is a solution of equation (1.2). If $y(T) = 0$ in G , P -a.s., then $y(t) = 0$ in G , P -a.s. for all $t \in [0, T]$.*

Remark 1.2 *The uniqueness problems for the solutions of both deterministic and stochastic partial differential equations have been studied for a long time. There are a great many positive results and some negative results. In case of time reversible systems, the backward uniqueness is equivalent to the classical (forward) uniqueness. If one considers the time irreversible system such as parabolic equations, the situation is quite different. The backward uniqueness implies the classical (forward) uniqueness, however, generally speaking, the converse conclusion is untrue.*

On account of the plentiful applications, such as studying the long time behavior of solutions and establishing the approximate controllability from the null controllability, the backward uniqueness for parabolic equations draws lots of attention (see [9, 10, 18, 19, 21] and the references cited therein). It is well understood now. On the contrast, as far as we know, [5] is the only paper concerned with backward uniqueness for stochastic parabolic equations in the literature. In [5], the authors obtained the backward uniqueness for semilinear stochastic parabolic equations with deterministic coefficients. They employed some deep tools in Stochastic Analysis to establish the result. However, it seems that their method depends on the very fact that the coefficients are deterministic and one cannot simply mimic their method to obtain Corollary 1.1, since the coefficients are random.

The other inverse problem studied in this paper is about the global uniqueness of an inverse source problem for stochastic parabolic equations. We first give a precise formulation of the problem.

Let $x = (x_1, x') \in \mathbb{R}^n$ and $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. Consider a special G as $G = (0, l) \times G'$, where $G' \subset \mathbb{R}^{n-1}$ be a bounded domain with a C^2 boundary. We consider the following stochastic parabolic equation:

$$\begin{cases} dy - \Delta y = [(b_1, \nabla y) + b_2 y + h(t, x')R(t, x)]dt + b_3 y dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = 0 & \text{in } G. \end{cases} \quad (1.6)$$

Here

$$b_1 \in L_{\mathcal{F}}^{\infty}(0, T; W^{1,\infty}(G; \mathbb{R}^n)), \quad b_2 \in L_{\mathcal{F}}^{\infty}(0, T; W^{1,\infty}(G)), \quad b_3 \in L_{\mathcal{F}}^{\infty}(0, T; W^{2,\infty}(G)),$$

and

$$R \in C^2([0, T] \times \overline{G}), \quad h \in L_{\mathcal{F}}^2(0, T; H^1(G)).$$

The inverse source problem studied here is as follows:

Let R be given and $t_0 > 0$. Determine the source function $h(t, x')$, $(t, x') \in (0, t_0) \times G'$, by means of the observation of $\frac{\partial y}{\partial \nu} \Big|_{[0, t_0] \times \partial G}$.

Here $\nu = (\nu^1, \dots, \nu^n) \in \mathbb{R}^n$ is the outer normal vector of Γ .

We have the following uniqueness result about the above problem.

Theorem 1.2 Let $y \in L_{\mathcal{F}}^2(\Omega; C([0, T]; H_0^1(G)))$, $y_{x_1} \in L_{\mathcal{F}}^2(\Omega; C([0, T]; H_0^1(G)))$ and

$$|R(t, x)| \neq 0 \text{ for all } (t, x) \in [0, t_0] \times \overline{G}. \quad (1.7)$$

If

$$\frac{\partial y}{\partial \nu} = 0 \text{ on } [0, t_0] \times \partial G, \quad P\text{-a.s.},$$

then

$$h(t, x') = 0 \text{ for all } (t, x') \in [0, t_0] \times G', \quad P\text{-a.s.}$$

Remark 1.3 *One can follow the proof of Theorem 1.2 to show the same conclusion of Theorem 1.2 for equation (1.2). Here we consider equation (1.6) for the sake of presenting the key idea in a simple way.*

In practical problems, it is important to specialize some proper data so that the parameter to be reconstructed is uniquely identifiable. In our model, the data utilized is the boundary normal derivative of the solution. This type of inverse problem is important in many branches of engineering sciences. For examples, an accurate estimation of a pollution source in a river, a determination of magnitude of groundwater pollution sources.

Remark 1.4 *In the literature, determining a spacewise dependent source function for parabolic equations has been considered comprehensively (see [6, 8, 13, 14, 24] and the references cited therein). A classical result for the deterministic setting is as follows.*

Consider the following parabolic equation:

$$\begin{cases} y_t - \Delta y = c_1 \nabla y + c_2 y + Rf & \text{in } Q, \\ y = 0 & \text{on } \Sigma. \end{cases} \quad (1.8)$$

Here c_1 and c_2 are suitable functions on Q . $R \in L^\infty(Q)$, $R_t \in L^\infty(Q)$ and $R(t_0, x) \neq 0$ in \overline{G} for some $t_0 \in (0, T]$. $f \in L^2(G)$ is independent of t . The authors in [12] proved the following result:

Assume that $y \in H^{2,1}(Q)$ and $y_t \in H^{2,1}(Q)$, then there exists a constant $C > 0$ such that

$$|f|_{L^2(G)} \leq C \left(|y(t_0)|_{H^2(G)} + \left| \frac{\partial y_t}{\partial \nu} \right|_{L^2(0,T;L^2(\Gamma_0))} \right), \quad (1.9)$$

where Γ_0 is any open subset of Γ .

Compared with Theorem 1.2, inequality (1.9) gives an explicit estimate for the source term by $|y(t_0)|_{H^2(G)}$ and $\left| \frac{\partial y_t}{\partial \nu} \right|_{L^2(0,T;L^2(\Gamma_0))}$. A key step in the proof of equality (1.9) is to differentiate the solution of (1.8) with respect to t . Unfortunately, the solution of (1.6) does not enjoy differentiability with respect to t . This leads to the difficulty to follow the proof for inequality (1.9) to solve our problem.

There are abundant works addressing to the inverse problems for PDEs. And it is even impossible to list the related papers owing to the big amount. Unfortunately, there exist a very few works addressing inverse problems for stochastic PDEs (See [3, 7, 11] for example). Although there are some people considering the inverse source problem for parabolic equations with random noise in the measurement (see [15] for example), to the best of our knowledge, there is no paper considering the inverse problem for stochastic parabolic equations.

Remark 1.5 *As we have pointed out in Remark 1.1 and Remark 1.4, the non-differentiability with respect to the variable with noise (say, the time variable considered in this paper) of the solution of a stochastic PDE usually leads substantially new difficulties in the study of inverse*

problems for stochastic PDEs. Another trouble for studying the inverse problem of stochastic PDEs is that the usual compactness embedding result does not remain true for the solution spaces related to stochastic PDEs. Due to these new phenomenons, some useful methods for solving inverse problems for deterministic PDEs (see [13, 17] for example) cannot be used to solve the corresponding inverse problems in the stochastic setting.

In this paper, we prove Theorem 1.1 and Theorem 1.2 by establishing a suitable Carleman estimate for stochastic parabolic equations. Consider the following equation:

$$\begin{cases} dy - \sum_{i,j=1}^n (b^{ij} y_i)_j dt = [(a_1, \nabla y) + a_2 y + f] dt + (a_3 y + g) dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \end{cases} \quad (1.10)$$

where $f \in L^2_{\mathcal{F}}(0, T; L^2(G))$ and $g \in L^2_{\mathcal{F}}(0, T; H^1(G))$.

Remark 1.6 Obviously, both equation (1.2) and equation (1.6) are special examples of equation (1.10).

To start with, we introduce some functions. Let $\psi \in C^\infty(\mathbb{R})$ with $|\psi_t| \geq 1$, which is independent of the x -variable. Put

$$\varphi = e^{\lambda\psi} \quad \text{and} \quad \theta = e^{s\varphi}. \quad (1.11)$$

We have the following result.

Theorem 1.3 Let $\delta \in [0, T)$. For all $y \in L^2_{\mathcal{F}}(\Omega; C([0, T]; H^1_0(G)))$ solve equation (1.10), there exists a $\lambda_1 > 0$ such that for all $\lambda \geq \lambda_1$, there exists an $s_0(\lambda_1) > 0$ so that for all $s \geq s_0(\lambda_1)$, it holds that

$$\begin{aligned} & \lambda \mathbb{E} \int_{\delta}^T \int_G \theta^2 |\nabla y|^2 dx dt + s \lambda^2 \mathbb{E} \int_{\delta}^T \int_G \varphi \theta^2 y^2 dx dt \\ & \leq C \mathbb{E} \left[\theta^2(T) |\nabla y(T)|_{L^2(G)}^2 + \theta^2(\delta) |\nabla y(\delta)|_{L^2(G)}^2 + s \lambda \varphi(T) \theta^2(T) |y(T)|_{L^2(G)}^2 \right. \\ & \quad \left. + s \lambda \varphi(\delta) \theta^2(\delta) |y(\delta)|_{L^2(G)}^2 + \int_{\delta}^T \int_G (f^2 + g^2 + |\nabla g|^2) dx dt \right]. \end{aligned} \quad (1.12)$$

Here and in the sequel, the constant C depends only on G , $(b^{ij})_{n \times n}$, T , δ and ψ , which may change from line to line.

Remark 1.7 It is well known that the global Carleman estimate is an important tool for the study of inverse problems for deterministic PDEs. Such kind of estimate has been introduced to solving inverse problems in [4], and were comprehensively studied in [13, 17]. Now it is a useful methodology for solving inverse problems (e.g. [13, 16, 17, 23, 24]).

Although there are numerous results for the global Carleman estimate for deterministic PDEs, people know very little about the stochastic counterpart. In fact, as far as we know,

[2, 22, 25] are the only three published papers addressing to the global Carleman estimate for stochastic PDEs. In [2, 22], some Carleman-type inequalities were established, for the sake of deriving the null controllability of stochastic parabolic equations. Note further that the weight function θ used in this paper (which plays a key role in the sequel) is quite different from that in [2, 22]. It seems that the Carleman estimate in [2, 22] cannot be applied to our problems. Indeed, the weight function θ in [2, 22] is supposed to vanish at 0 and T , and therefore it does not serve the purpose of proving Theorem 1.1 and Theorem 1.2.

The rest paper is organized as follows. In Section 2, we prove Theorem 1.3. Section 3 is addressed to the proof of Theorem 1.1. At last, in Section 4, we give a proof for Theorem 1.2.

2 Carleman estimate for stochastic parabolic equations

In this section, we give a proof of Theorem 1.3.

We first give a weighted identity, which will play an important role in the sequel.

Proposition 2.1 *Assume that u is an $H^2(\mathbb{R}^n)$ -valued continuous semi-martingale. Put $v = \theta u$, then we have the following equality:*

$$\begin{aligned}
& -\theta \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s\lambda\varphi\psi_t v \right] \left[du - \sum_{i,j=1}^n (b^{ij} u_{x_i})_{x_j} dt \right] + \frac{1}{4}\lambda\theta v \left[du - \sum_{i,j=1}^n (b^{ij} u_{x_i})_{x_j} dt \right] \\
& = - \sum_{i,j=1}^n \left(b^{ij} v_{x_i} dv + \frac{1}{4} b^{ij} v_{x_i} v dt \right)_{x_j} + \frac{1}{2} d \left(\sum_{i,j=1}^n b^{ij} v_{x_i} v_{x_j} - s\lambda\varphi\psi_t v^2 + \frac{1}{8}\lambda v^2 \right) \\
& \quad - \left(\frac{1}{2} \sum_{i,j=1}^n b^{ij} dv_{x_i} dv_{x_j} + \frac{1}{2} \sum_{i,j=1}^n b_t^{ij} v_{x_i} v_{x_j} dt - \frac{1}{4}\lambda \sum_{i,j=1}^n b^{ij} v_{x_i} v_{x_j} dt \right) \\
& \quad + \frac{1}{2} s\lambda^2 \varphi \psi_t^2 v^2 dt + \frac{1}{2} s\lambda \varphi \psi_{tt} v^2 dt + \frac{1}{2} s\lambda \varphi \psi_t (dv)^2 - \frac{1}{8} \lambda (dv)^2 \\
& \quad + \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s\lambda\varphi v \right]^2 dt.
\end{aligned} \tag{2.1}$$

Proof: The proof is based on some direct computation by Itô's stochastic calculus. The

first term in the left hand side of equality (2.1) reads as

$$\begin{aligned}
& -\theta \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s\lambda\varphi\psi_t v \right] \left[du - \sum_{i,j=1}^n (b^{ij} u_{x_i})_{x_j} dt \right] \\
&= - \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s\lambda\varphi\psi_t v \right] \left[dv - \sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} dt - s\lambda\varphi\psi_t v dt \right] \\
&= - \sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} dv - s\lambda\varphi\psi_t v dv + \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s\lambda\varphi v \right]^2 dt \\
&= - \sum_{i,j=1}^n (b^{ij} v_{x_i} dv)_{x_j} + \frac{1}{2} \sum_{i,j=1}^n d(b^{ij} v_{x_i} v_{x_j}) - \frac{1}{2} \sum_{i,j=1}^n b^{ij} dv_{x_i} dv_{x_j} \\
&\quad - \frac{1}{2} \sum_{i,j=1}^n b_t^{ij} v_{x_i} v_{x_j} dt - \frac{1}{2} d(s\lambda\varphi\psi_t v^2) + \frac{1}{2} s\lambda^2 \psi_t^2 \varphi v^2 dt \\
&\quad + \frac{1}{2} s\lambda\varphi\psi_{tt} v^2 dt + \frac{1}{2} s\lambda\psi_t \varphi (dv)^2 + \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s\lambda\varphi v \right]^2 dt.
\end{aligned} \tag{2.2}$$

The second term in the left hand side of equality (2.1) satisfies

$$\begin{aligned}
& \frac{1}{4} \lambda \theta v \left[du - \sum_{i,j=1}^n (b^{ij} u_{x_i})_{x_j} dt \right] \\
&= \frac{1}{4} \lambda v \left[dv - \sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} dt - s\lambda\varphi\psi_t v dt \right] \\
&= \frac{1}{4} \lambda v dv - \frac{1}{4} \lambda v \sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} dt - \frac{1}{4} s\lambda^2 \varphi\psi_t v^2 dt \\
&= \frac{1}{8} \lambda dv^2 - \frac{1}{8} \lambda (dv)^2 - \frac{1}{4} \lambda \sum_{i,j=1}^n (b^{ij} v_{x_i} v)_{x_j} dt + \frac{1}{4} \lambda \sum_{i,j=1}^n b^{ij} v_{x_i} v_{x_j} dt - \frac{1}{4} s\lambda^2 \psi_t \varphi v^2 dt.
\end{aligned} \tag{2.3}$$

This, together with equality (2.2), implies equality (2.1). \square

Now we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3: Applying Proposition 2.1 to equation (1.10) with $u = y$, integrating equality (2.1) on $[\delta, T] \times G$ for some $\delta \in [0, T)$, and taking mathematical expectation,

we get that

$$\begin{aligned}
& -\mathbb{E} \int_{\delta}^T \int_G \theta \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s\lambda \varphi \psi_t v \right] \left[du - \sum_{i,j=1}^n (b^{ij} u_{x_i})_{x_j} dt \right] dx \\
& + \frac{1}{4} \lambda \mathbb{E} \int_{\delta}^T \int_G \theta v \left[du - \sum_{i,j=1}^n (b^{ij} u_{x_i})_{x_j} dt \right] dx \\
& = -\mathbb{E} \int_{\delta}^T \int_G \sum_{i,j=1}^n \left(b^{ij} v_{x_i} dv + \frac{1}{4} \lambda b^{ij} v_{x_i} v dt \right)_{x_j} dx + \frac{1}{2} \mathbb{E} \int_{\delta}^T \int_G d \left(\sum_{i,j=1}^n b^{ij} v_{x_i} v_{x_j} - s\lambda \varphi \psi_t v^2 + \frac{1}{8} \lambda v^2 \right) dx \\
& + \mathbb{E} \int_{\delta}^T \int_G \left(\frac{1}{4} \lambda \sum_{i,j=1}^n b^{ij} v_{x_i} v_{x_j} dt + \frac{1}{2} \sum_{i,j=1}^n b_t^{ij} v_{x_i} v_{x_j} dt - \frac{1}{2} \sum_{i,j=1}^n b^{ij} dv_{x_i} dv_{x_j} \right) dx \\
& + \mathbb{E} \int_{\delta}^T \int_G \left[\frac{1}{2} s \lambda^2 \varphi \psi_t^2 v^2 dt + \frac{1}{2} s \lambda \varphi \psi_{tt} v^2 dt - \frac{1}{4} s \lambda^2 \varphi \psi_t v^2 dt + \frac{1}{2} s \lambda \varphi \psi_t (dv)^2 - \frac{1}{8} \lambda (dv)^2 \right] dx \\
& + \mathbb{E} \int_{\delta}^T \int_G \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s\lambda \varphi \psi_t v \right]^2 dx dt.
\end{aligned} \tag{2.4}$$

Now we estimate the terms in the right hand side of equality (2.4) one by one.

For the first one, since $y|_{\Sigma} = 0$, we have that $v|_{\Sigma} = 0$. Therefore, it holds that

$$\begin{aligned}
& -\mathbb{E} \int_{\delta}^T \int_G \sum_{i,j} \left(b^{ij} v_{x_i} dv + \frac{1}{4} \lambda b^{ij} v_{x_i} v dt \right)_{x_j} dx \\
& = -\mathbb{E} \int_{\delta}^T \int_{\Gamma} \sum_{i,j=1}^n \left(b^{ij} v_{x_i} dv + \frac{1}{4} \lambda b^{ij} v_{x_i} v dt \right) \nu^j d\Gamma = 0.
\end{aligned} \tag{2.5}$$

For the second one, we have

$$\begin{aligned}
& \frac{1}{2} \mathbb{E} \int_{\delta}^T \int_G d \left(\sum_{i,j=1}^n b^{ij} v_{x_i} v_{x_j} - s\lambda \varphi \psi_t v^2 + \frac{1}{8} \lambda v^2 \right) dx \\
& \geq -C \mathbb{E} \left(|\nabla v(T)|_{L^2(G)}^2 + |\nabla v(\delta)|_{L^2(G)}^2 + s\lambda \varphi(T) |v(T)|_{L^2(G)}^2 + s\lambda \varphi(\delta) |v(\delta)|_{L^2(G)}^2 \right).
\end{aligned} \tag{2.6}$$

The third one reads as

$$\begin{aligned}
& \mathbb{E} \int_{\delta}^T \int_G \left(\frac{1}{4} \lambda \sum_{i,j=1}^n b^{ij} v_{x_i} v_{x_j} dt + \frac{1}{2} \sum_{i,j=1}^n b_t^{ij} v_{x_i} v_{x_j} dt - \frac{1}{2} \sum_{i,j=1}^n b^{ij} dv_{x_i} dv_{x_j} \right) dx \\
& \geq \mathbb{E} \int_{\delta}^T \int_G \left[\frac{1}{4} \lambda \sigma |\nabla v|^2 - C |\nabla v|^2 - C (a_3^2 |\nabla v|^2 + |\nabla a_3|^2 v^2 + |\nabla g|^2 + |g|^2) \right] dx dt \\
& \geq \frac{1}{4} \lambda \mathbb{E} \int_{\delta}^T \int_G \sigma |\nabla v|^2 dx dt - C (|a_3|_{L^\infty(0,T;W^{1,\infty}(G))}^2 + 1) \mathbb{E} \int_{\delta}^T \int_G (|\nabla v|^2 + v^2) dx dt \\
& \quad - C \mathbb{E} \int_{\delta}^T \int_G (|\nabla g|^2 + g^2) dx dt.
\end{aligned} \tag{2.7}$$

For the forth one, recalling that $|\psi_t| \geq 1$, we see

$$\begin{aligned} & \mathbb{E} \int_{\delta}^T \int_G \left[\frac{1}{2} s \lambda^2 \varphi \psi_t^2 v^2 dt + \frac{1}{2} s \lambda \varphi \psi_{tt} v^2 dt - \frac{1}{4} s \lambda^2 \varphi \psi_t v^2 dt + \frac{1}{2} s \lambda \varphi \psi_t (dv)^2 - \frac{1}{8} \lambda (dv)^2 \right] dx \\ & \geq \frac{1}{4} s \lambda^2 \mathbb{E} \int_{\delta}^T \int_G \varphi v^2 dx dt + s O(\lambda) \mathbb{E} \int_{\delta}^T \int_G \varphi v^2 dx dt. \end{aligned} \quad (2.8)$$

Thus, we know that there exists a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, it holds that

$$\begin{aligned} & \mathbb{E} \int_{\delta}^T \int_G \left[\frac{1}{2} s \lambda^2 \varphi \psi_t^2 v^2 dt + \frac{1}{2} s \lambda \varphi \psi_{tt} v^2 dt - \frac{1}{4} s \lambda^2 \varphi \psi_t v^2 dt + \frac{1}{2} s \lambda \varphi \psi_t (dv)^2 - \frac{1}{8} \lambda (dv)^2 \right] dx \\ & \geq \frac{1}{8} s \lambda^2 \mathbb{E} \int_{\delta}^T \int_G \varphi v^2 dx dt. \end{aligned} \quad (2.9)$$

Now, we estimate the terms in the left hand side one by one. By equation (1.2), we know that

$$\begin{aligned} & -\mathbb{E} \int_{\delta}^T \int_G \theta \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s \lambda \varphi v \right] \left[du - \sum_{i,j=1}^n (b^{ij} u_{x_i})_{x_j} dt \right] dx \\ & \leq \mathbb{E} \int_{\delta}^T \int_G \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s \lambda \varphi v \right]^2 dx dt + 3 \mathbb{E} \int_{\delta}^T \int_G \theta^2 (|a_1|^2 |\nabla u|^2 + a_2^2 u^2 + f^2) dx dt \\ & \leq \mathbb{E} \int_{\delta}^T \int_G \left[\sum_{i,j=1}^n (b^{ij} v_{x_i})_{x_j} + s \lambda \varphi v \right]^2 dx dt + 3 |a_1|_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G;\mathbb{R}^n))}^2 \mathbb{E} \int_{\delta}^T \int_G |\nabla v|^2 dx dt \\ & \quad + 3 |a_2|_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G))}^2 \mathbb{E} \int_{\delta}^T \int_G v^2 dx dt + 3 \mathbb{E} \int_{\delta}^T \int_G f^2 dx dt, \end{aligned} \quad (2.10)$$

and that

$$\begin{aligned} & \frac{1}{4} \lambda \mathbb{E} \int_{\delta}^T \int_G \theta v \left[du - \sum_{i,j=1}^n (b^{ij} u_{x_i})_{x_j} dt \right] dx \\ & \leq \frac{1}{64} \lambda^2 \mathbb{E} \int_{\delta}^T \int_G v^2 dx dt + 3 \mathbb{E} \int_{\delta}^T \int_G \theta^2 (|a_1|^2 |\nabla u|^2 + a_2^2 u^2 + f^2) dx dt \\ & \leq \frac{1}{64} \lambda^2 \mathbb{E} \int_{\delta}^T \int_G v^2 dx dt + 3 |a_1|_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G;\mathbb{R}^n))}^2 \mathbb{E} \int_{\delta}^T \int_G |\nabla v|^2 dx dt \\ & \quad + 3 |a_2|_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G))}^2 \mathbb{E} \int_{\delta}^T \int_G v^2 dx dt + 3 \mathbb{E} \int_{\delta}^T \int_G f^2 dx dt. \end{aligned} \quad (2.11)$$

From (2.4)-(2.11), we find

$$\begin{aligned}
& \frac{1}{4}\lambda\mathbb{E}\int_{\delta}^T\int_G|\nabla v|^2dxdt - C(|a_1|_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G;\mathbb{R}^n))}^2 + |a_3|_{L^{\infty}(0,T;W^{1,\infty}(G))}^2 + 1)\mathbb{E}\int_{\delta}^T\int_G|\nabla v|^2dxdt \\
& - C(|a_2|_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G))}^2 + |a_3|_{L^{\infty}(0,T;W^{1,\infty}(G))}^2 + 1)\mathbb{E}\int_{\delta}^T\int_G|\nabla v|^2dxdt \\
& + \left(\frac{1}{8}s\lambda^2 - \frac{1}{64}\lambda^2\right)\mathbb{E}\int_{\delta}^T\int_G\varphi v^2dxdt \\
& \leq C\mathbb{E}\left[|\nabla v(T)|_{L^2(G)}^2 + |\nabla v(\delta)|_{L^2(G)}^2 + s\lambda\varphi(T)|v(T)|_{L^2(G)}^2 + s\lambda\varphi(\delta)|v(\delta)|_{L^2(G)}^2\right. \\
& \quad \left. + \int_{\delta}^T\int_G(f^2 + g^2 + |\nabla g|^2)dxdt\right].
\end{aligned} \tag{2.12}$$

Let

$$r_1 = |a_1|_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G;\mathbb{R}^n))}^2 + |a_2|_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G))}^2 + |a_3|_{L^{\infty}(0,T;W^{1,\infty}(G))}^2 + 1,$$

from inequality (2.12), we know that there exists a $\lambda_1 \geq \max\{Cr_1, \lambda_0\}$ such that for all $\lambda \geq \lambda_1$, there exists a $s_0(\lambda_1) > 0$ so that for all $s \geq s_0(\lambda_1)$, it holds that

$$\begin{aligned}
& \lambda\mathbb{E}\int_{\delta}^T\int_G|\nabla v|^2dxdt + s\lambda^2\mathbb{E}\int_{\delta}^T\int_G\varphi v^2dxdt \\
& \leq C\mathbb{E}\left[|\nabla v(T)|_{L^2(G)}^2 + |\nabla v(\delta)|_{L^2(G)}^2 + s\lambda\varphi(T)|v(T)|_{L^2(G)}^2 + s\lambda\varphi(\delta)|v(\delta)|_{L^2(G)}^2\right. \\
& \quad \left. + \int_{\delta}^T\int_G(f^2 + g^2 + |\nabla g|^2)dxdt\right],
\end{aligned} \tag{2.13}$$

which implies inequality (1.12) immediately. \square

3 Proof for Theorem 1.1

This section is devoted to proving Theorem 1.1. We borrow some ideas from [24].

Proof of Theorem 1.1: Choose t_1 and t_2 such that $0 < t_1 < t_2 < t_0$. Set $\alpha_k = e^{\lambda t_k}$ ($k = 0, 1, 2$). Let $\rho \in C^{\infty}(\mathbb{R})$ such that $0 \leq \rho \leq 1$ and that

$$\rho = \begin{cases} 1, & t \geq t_2, \\ 0, & t \leq t_1. \end{cases} \tag{3.1}$$

Let $z = \chi y$, by means of y solves equation (1.2), we know that z solves

$$\begin{cases} dz - \sum_{i,j=1}^n (b^{ij} z_{x_i})_{x_j} dt = [(a_1, \nabla z) + a_2 z + \rho_t(t)y] dt + a_3 z dB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = 0 & \text{in } G. \end{cases} \tag{3.2}$$

Applying Theorem 1.3 with $\psi = t$ and $\delta = 0$ to equation (3.2), for $\lambda \geq \lambda_1$ and $s \geq s_0(\lambda_1)$, we have

$$\begin{aligned} & \lambda \mathbb{E} \int_Q \theta^2 |\nabla z|^2 dx dt + s \lambda^2 \mathbb{E} \int_Q \theta^2 \varphi |z|^2 dx dt \\ & \leq C \mathbb{E} \left[\theta^2(T) |\nabla z(T)|_{L^2(G)}^2 + s \lambda \varphi(T) \theta^2(T) |z(T)|_{L^2(G)}^2 + \int_Q \theta^2 |\rho_t(t) y|^2 dx dt \right]. \end{aligned} \quad (3.3)$$

By virtue of (3.1), we see that

$$\mathbb{E} \int_Q \theta^2 |\rho_t(t)|^2 y^2 dx dt \leq C \int_{t_2}^{t_1} \int_G \theta^2 y^2 dx dt \leq C \theta^2(t_1) |y|_{L_{\mathcal{F}}^2(0,T;L^2(G))}^2. \quad (3.4)$$

This, together with inequality (3.3), implies that

$$\begin{aligned} & \lambda \theta^2(t_0) \mathbb{E} \int_{t_0}^T \int_G |\nabla y|^2 dx dt + s \lambda^2 \theta^2(t_0) \mathbb{E} \int_{t_0}^T \int_G \varphi |y|^2 dx dt \\ & \leq \lambda \mathbb{E} \int_Q \theta^2 |\nabla z|^2 dx dt + s \lambda^2 \mathbb{E} \int_Q \theta^2 \varphi |z|^2 dx dt \\ & \leq C \theta^2(t_1) |y|_{L_{\mathcal{F}}^2(0,T;L^2(G))}^2 + C \mathbb{E} \left(\theta^2(T) |\nabla y(T)|_{L^2(G)}^2 + s \lambda \varphi(T) \theta^2(T) |y(T)|_{L^2(G)}^2 \right). \end{aligned} \quad (3.5)$$

Here we utilizing the fact that $\theta(t) \leq \theta(s)$ for $t \leq s$.

From inequality (3.5), we see

$$\begin{aligned} & \lambda \mathbb{E} \int_{t_0}^T \int_G |\nabla y|^2 dx dt + s \lambda^2 \mathbb{E} \int_{t_0}^T \int_G \varphi |y|^2 dx dt \\ & \leq C \theta^2(t_1) \theta^{-2}(t_0) |y|_{L_{\mathcal{F}}^2(0,T;L^2(G))}^2 + C \mathbb{E} \left(\theta^2(T) |\nabla y(T)|_{L^2(G)}^2 + s \lambda \varphi(T) \theta^2(T) |y(T)|_{L^2(G)}^2 \right). \end{aligned} \quad (3.6)$$

By means of $dy^2 = 2ydy + (dy)^2$, we obtain that

$$\begin{aligned} & \mathbb{E} \int_G |y(t_0)|^2 dx \\ & = \mathbb{E} \int_G |y(T)|^2 dx - \mathbb{E} \int_{t_0}^T \int_G [2ydy + (dy)^2] dx \\ & = \mathbb{E} \int_G |y(T)|^2 dx - \mathbb{E} \int_{t_0}^T \int_G \left\{ 2y \left[\sum_{i,j=1}^n (b^{ij} y_{x_i})_{x_j} + (a_1, \nabla y) + a_2 y \right] + (a_3 y)^2 \right\} dx dt \\ & \leq \mathbb{E} \int_G |y(T)|^2 dx + C \mathbb{E} \int_{t_0}^T \int_G |\nabla y|^2 dx dt \\ & \quad + (|a_1|_{L^\infty(0,T;L^\infty(G;\mathbb{R}^n))}^2 + |a_2|_{L^\infty(0,T;L^\infty(G))} + |a_2|_{L^\infty(0,T;L^\infty(G))}^2) \mathbb{E} \int_{t_0}^T \int_G y^2 dx dt \\ & \leq \mathbb{E} \int_G |y(T)|^2 dx + C \mathbb{E} \int_{t_0}^T \int_G |\nabla y|^2 dx dt + C r_1 \mathbb{E} \int_{t_0}^T \int_G y^2 dx dt, \end{aligned} \quad (3.7)$$

Recalling $\varphi \geq 1$, from inequality (3.7), we know that there exists a $\lambda_2 > 0$ such that for all $\lambda \geq \lambda_2$, it holds that

$$\begin{aligned} & \mathbb{E} \int_G |y(t_0)|^2 dx \\ & \leq \mathbb{E} \int_G |y(T)|^2 dx + C \left(\lambda \mathbb{E} \int_{t_0}^T \int_G |\nabla y|^2 dx dt + s \lambda^2 \mathbb{E} \int_{t_0}^T \int_G \varphi y^2 dx dt \right). \end{aligned} \quad (3.8)$$

Combing inequality (3.6) and inequality (3.8), for any $\lambda \geq \max\{\lambda_1, \lambda_2\}$ and $s \geq s_0(\lambda_1)$, we have

$$\begin{aligned} & \mathbb{E} \int_G |y(t_0)|^2 dx \\ & \leq C \theta^2(t_1) \theta^{-2}(t_0) |y|_{L^2_{\mathcal{F}}(0,T;L^2(G))}^2 + C \mathbb{E} \left(\theta^2(T) |\nabla y(T)|_{L^2(G)}^2 + s \lambda \varphi(T) \theta^2(T) |y(T)|_{L^2(G)}^2 \right). \end{aligned} \quad (3.9)$$

Now we fix $\lambda_3 = \max\{\lambda_1, \lambda_2\}$, from inequality (3.9), we get

$$\mathbb{E} \int_G |y(t_0)|^2 dx \leq C \theta^2(t_1) \theta^{-2}(t_0) |y|_{L^2_{\mathcal{F}}(0,T;L^2(G))}^2 + C \theta^2(T) \mathbb{E} |y(T)|_{H^1(G)}^2. \quad (3.10)$$

Replacing C by $C e^{s_0 e^{\lambda_3 T}}$, from inequality (3.10), for any $s > 0$, it holds that

$$\mathbb{E} \int_G |y(t_0)|^2 dx \leq C e^{-2s(e^{\lambda_3 t_1} - e^{\lambda_3 t_0})} |y|_{L^2_{\mathcal{F}}(0,T;L^2(G))}^2 + C e^{Cs} \mathbb{E} |y(T)|_{H^1(G)}^2. \quad (3.11)$$

Choosing $s \geq 0$ minimizing the right-hand side of inequality (3.11), we obtain that

$$\mathbb{E} |y(t_0)|_{L^2(G)}^2 \leq C |y|_{L^2_{\mathcal{F}}(0,T;L^2(G))}^{\frac{C}{C+2(e^{\lambda_3 t_0} - e^{\lambda_3 t_1})}} \mathbb{E} |y(T)|_{H^1(G)}^{\frac{2(e^{\lambda_3 t_0} - e^{\lambda_3 t_1})}{C+2(e^{\lambda_3 t_0} - e^{\lambda_3 t_1})}}, \quad (3.12)$$

which implies inequality (1.3) immediately.

4 Proof of Theorem 1.2

This section is addressed to proving Theorem 1.2. We borrow some ideas in [24] again.

Proof of Theorem 1.2: For arbitrary small $\varepsilon > 0$, we choose t_1 and t_2 such that

$$0 < t_0 - \varepsilon < t_1 < t_2 < t_0.$$

Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function such that $0 \leq \chi \leq 1$ and that

$$\chi = \begin{cases} 1, & t \leq t_1, \\ 0, & t \geq t_2. \end{cases} \quad (4.1)$$

Put $y = Rz$ (recall (1.7) for R) in $[0, t_2] \times G$, by virtue of y is a solution of equation (1.6), we know that z solves

$$\left\{ \begin{array}{ll} dz - \Delta z dt = \left[(b_1, \nabla z) + \left(\frac{2\nabla R}{R}, \nabla z \right) + \left(b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} \right. \right. \\ \quad \left. \left. - \frac{R_t}{R} + \left(\frac{\nabla R}{R}, b_1 \right) \right) z \right] dt + h dt + b_3 z dB(t) & \text{in } [0, t_0] \times G, \\ z = \frac{\partial z}{\partial \nu} = 0 & \text{on } [0, t_0] \times \Gamma, \\ z(0) = 0 & \text{in } G. \end{array} \right. \quad (4.2)$$

Differentiating both sides of equation (4.2) and setting $u = z_{x_1}$, we obtain that

$$\left\{ \begin{array}{ll} du - \Delta u dt = \left[((b_1)_{x_1}, \nabla z) + (b_1, \nabla u) + \left(\left(\frac{2\nabla R}{R} \right)_{x_1}, \nabla z \right) + \left(\frac{2\nabla R}{R}, \nabla u \right) \right. \\ \quad + \left(b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} - \frac{R_t}{R} + \left(\frac{\nabla R}{R}, b_1 \right) \right)_{x_1} z \\ \quad + \left(b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} - \frac{R_t}{R} + \left(\frac{\nabla R}{R}, b_1 \right) \right) u \left. \right] dt \\ \quad + (b_3)_{x_1} z dt + b_3 u dB(t) & \text{in } [0, t_0] \times G, \\ u = 0 & \text{on } [0, t_0] \times \Gamma \\ u(0) = 0 & \text{in } G. \end{array} \right. \quad (4.3)$$

Set $w = \chi u$, then we know that w solves the following equation

$$\left\{ \begin{array}{ll} dw - \Delta w dt = \left[((b_1)_{x_1}, \chi \nabla z) + (b_1, \nabla w) + \left(\left(\frac{2\nabla R}{R} \right)_{x_1}, \chi \nabla z \right) + \left(\frac{2\nabla R}{R}, \nabla w \right) \right. \\ \quad + \left(b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} - \frac{R_t}{R} + \left(\frac{\nabla R}{R}, b_1 \right) \right)_{x_1} \chi z \\ \quad + \left(b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} - \frac{R_t}{R} + \left(\frac{\nabla R}{R}, b_1 \right) \right) w \left. \right] dt \\ \quad + (b_3)_{x_1} \chi z dB(t) + b_3 w dB(t) - \chi' u dt & \text{in } [0, t_0] \times G, \\ w = 0 & \text{on } [0, t_0] \times \Gamma. \end{array} \right. \quad (4.4)$$

By means of $u = z_{x_1}$ and $z(t, 0, x') = y(t, 0, x') = 0$ for $(t, x') \in (0, t_0) \times G'$, we see

$$\chi z = \chi \int_0^{x_1} u(t, \eta, x') d\eta = \int_0^{x_1} w(t, \eta, x') d\eta. \quad (4.5)$$

This, together with equation (4.4), implies that v solves the following equation

$$\left\{ \begin{aligned} dw - \Delta w dt &= \left[(b_1, \nabla w) + \left(\frac{2\nabla R}{R}, \nabla w \right) + \left((b_1)_{x_1}, \nabla \int_0^{x_1} w(t, \eta, x') d\eta \right) \right. \\ &\quad + \left(\left(\frac{2\nabla R}{R} \right)_{x_1}, \nabla \int_0^{x_1} w(t, \eta, x') d\eta \right) \\ &\quad + \left(b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} - \frac{R_t}{R} + \left(\frac{\nabla R}{R}, b_1 \right) \right) w \\ &\quad + \left(b_2 + \frac{\Delta R}{R} - \frac{R_t}{R} + \left(\frac{\nabla R}{R}, b_1 \right) \right)_{x_1} \int_0^{x_1} w(t, \eta, x') d\eta \Big] dt \\ &\quad + (b_3)_{x_1} \chi \int_0^{x_1} u(t, \eta, x') d\eta dB(t) + b_3 w dB(t) - \chi' u dt \quad \text{in } [0, t_0] \times G, \\ w &= 0 \quad \text{on } [0, t_0] \times \Gamma. \end{aligned} \right. \quad (4.6)$$

Applying Theorem 1.3 to equation (4.6) with $\psi(t) = -t$, noting that $v(0) = \chi(0)u(0) = 0$ and $v(t_0) = \chi(t_0)u(t_0) = 0$, we get that

$$\begin{aligned} &\mathbb{E} \int_0^{t_0} \int_G \theta^2 (\lambda |\nabla w|^2 + s \lambda^2 w^2) dx dt \\ &\leq C \mathbb{E} \int_0^{t_0} \int_G \theta^2 |\chi' u|^2 dx dt \\ &\quad + C r_1 \mathbb{E} \int_0^{t_0} \int_G \theta^2 \left(\left| \int_0^{x_1} w(t, \eta, x') d\eta \right|^2 + \left| \int_0^{x_1} |\nabla w(t, \eta, x') d\eta| \right|^2 \right) dx dt. \end{aligned} \quad (4.7)$$

Since

$$\left| \int_0^{x_1} w(t, \eta, x') d\eta \right|^2 \leq l \int_0^l |w(t, \eta, x')|^2 d\eta,$$

we know

$$\begin{aligned} \int_0^{t_0} \int_G \theta^2 \left| \int_0^{x_1} w(t, \eta, x') d\eta \right|^2 dx dt &\leq l \int_0^l dx_1 \int_0^{t_0} \int_{G'} \int_0^l \theta^2 |w(t, \eta, x')|^2 d\eta dx' dt \\ &\leq l^2 \int_0^{t_0} \int_G \theta^2 |w(t, \eta, x')|^2 d\eta dx' dt. \end{aligned} \quad (4.8)$$

By virtue of

$$\nabla \int_0^{x_1} w(t, \eta, x') d\eta = \int_0^{x_1} \nabla w(t, \eta, x') d\eta + w(t, 0, x') = \int_0^{x_1} \nabla w(t, \eta, x') d\eta,$$

we get that

$$\begin{aligned} \int_0^{t_0} \int_G \theta^2 \left| \nabla \int_0^{x_1} w(t, \eta, x') d\eta \right|^2 dx dt &= \int_0^{t_0} \int_G \theta^2 \left| \nabla \int_0^{x_1} w(t, \eta, x') d\eta \right|^2 dx dt \\ &\leq l \int_0^l dx_1 \int_0^{t_0} \int_{G'} \int_0^l \theta^2 |\nabla w(t, \eta, x')|^2 d\eta dx' dt \\ &\leq l^2 \int_0^{t_0} \int_G \theta^2 |\nabla w(t, \eta, x')|^2 d\eta dx' dt. \end{aligned} \quad (4.9)$$

From inequality (4.7) – (4.9), we obtain that

$$\begin{aligned} & \mathbb{E} \int_0^{t_0} \int_G \theta^2 (\lambda |\nabla w|^2 + s \lambda^2 w^2) dx dt \\ & \leq C \mathbb{E} \int_0^{t_0} \int_G \theta^2 |\chi' u|^2 dx dt + C l^2 r_1 \mathbb{E} \int_0^{t_0} \int_G \theta^2 (|\nabla w|^2 + |w|^2) dx dt. \end{aligned} \quad (4.10)$$

Thus, we know that there is a $\lambda_4 = \max\{Cr_1, \lambda_1\}$ such that for all $\lambda \geq \lambda_4$, there exists an $s_1(\lambda_4) > 0$ so that for all $s \geq s_1(\lambda_4)$, it holds that

$$\mathbb{E} \int_0^{t_0} \int_G \theta^2 (\lambda |\nabla w|^2 + s \lambda^2 w^2) dx dt \leq C \mathbb{E} \int_0^{t_0} \int_G \theta^2 |\chi' u|^2 dx dt. \quad (4.11)$$

Fix $\lambda = \lambda_4$, by the property of χ (see (4.1)), we find

$$\mathbb{E} \int_0^{t_0} \int_G \theta^2 |\chi' u|^2 dx dt \leq e^{2se^{-\lambda_4 t_1}} \mathbb{E} \int_Q |u|^2 dx dt \leq e^{2se^{-\lambda_4 t_1}} |y_{x_1}|_{L^2_{\mathcal{F}}(0, T; L^2(G))}^2. \quad (4.12)$$

This, together with inequality (4.11), implies that for all $s \geq s_1$, it holds that

$$\begin{aligned} e^{2se^{-\lambda_4(t_0-\varepsilon)}} \mathbb{E} \int_0^{t_0-\varepsilon} \int_G (|\nabla w|^2 + sw^2) dx dt & \leq \mathbb{E} \int_0^{t_0-\varepsilon} \int_G \theta^2 (|\nabla w|^2 + sw^2) dx dt \\ & \leq \mathbb{E} \int_0^{t_0} \int_G \theta^2 (|\nabla w|^2 + sw^2) dx dt \\ & \leq C e^{2se^{-\lambda_4 t_1}} |y_{x_1}|_{L^2_{\mathcal{F}}(0, T; L^2(G))}^2. \end{aligned} \quad (4.13)$$

From inequality (4.13), we have

$$|w|_{L^2_{\mathcal{F}}(0, T; H^1(G))}^2 \leq C e^{2s(e^{-\lambda_4 t_1} - e^{-\lambda_4(t_0-\varepsilon)})} |y_{x_1}|_{L^2_{\mathcal{F}}(0, T; L^2(G))}^2. \quad (4.14)$$

Recalling that $t_0 - \varepsilon < t_1$, we know $e^{-\lambda_4 t_1} - e^{-\lambda_4(t_0-\varepsilon)} < 0$. Letting $s \rightarrow +\infty$, we obtain that

$$w = 0 \quad \text{in } (0, t_0 - \varepsilon) \times G, \quad P\text{-a.s.}$$

This, together with equality (4.5), implies that

$$z = 0 \quad \text{in } (0, t_0 - \varepsilon) \times G, \quad P\text{-a.s.},$$

which means

$$h = 0 \quad \text{in } (0, t_0 - \varepsilon) \times G', \quad P\text{-a.s.}$$

Since $\varepsilon > 0$ is arbitrary, the proof of Theorem 1.2 is completed. \square

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